MATH2068: Honours Mathematical Analysis II: Home Test; 2nd term

Starting: 5:00 pm, 4 April 2025

Important Notice:

♣ The answer paper **Must Be Submitted before 5:00 pm, 5 April 2025.**
♠ The answer paper MUST BE sent to the CU Blackboard.
♠ The answer paper MUST BE sent in pdf format IN ONE-file (Other format files, for example cell phone photos jpg files, are NOT ACCEPTED).
✠ The answer paper must include your name and student ID.

Answer ALL Questions

(1) **(20 points)**

Let (J_k) be a sequence of disjoint open intervals in $[0,1]$. Notice that for each n, $[0,1] \setminus \bigcup_{k=1}^{n} J_k$ can be written as the disjoint union of closed intervals $\bigcup_{i=0}^{n} F_{n,i}$. We assume that

(a) $\lim_n \max_{0 \leq i \leq n} |F_{n,i}| = 0$, where $|F_{n,i}|$ denotes the length of $F_{n,i}$;

(b) $\sum_{k=1}^{\infty} |J_k| := r < 1$.

Let (ϕ_k) be a sequence of continuous functions defined on $[0,1]$ satisfying $0 < \phi_k \leq \frac{1}{2^k}$ on J_k and $\phi_k \equiv 0$ on $[0,1] \setminus J_k$ for all k. Let $\phi := \sum_{k=1}^{\infty} \phi_k$. Define a function $h : [0,1] \to \mathbb{R}$ by

$$h(t) = \begin{cases} 1 & \phi(t) > 0 \\ 0 & \phi(t) = 0 \end{cases}$$

Show that $h \notin R[0,1]$.

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(2) **(20 points)**

Give an example of a sequence of disjoint open intervals (J_k) in $[0,1]$ satisfying the conditions (a) and (b) in Question (1) above.

From this, shows that there are $f \in R[0,1]$ and $\phi \in C[0,1]$ such that $f \circ \phi \notin R[0,1]$.

Note: we always have the fact that $f \circ \phi$ is Riemann integrable whenever f is continuous and ϕ is Riemann integrable (see the note).

$***$ **END OF PAPER** $***$